1 Quantum Stabilizer Codes

In this poster, we investigate the optimal [[n, k]] quantum stabilizer codes over the Pauli Independent X-Z Channel. By optimal code we mean

1. Such code achieves the smallest exact average error probability among all possible [[n, k]] quantum stabilizer codes
2. The average error probability is derived using the quantum maximal-likelihood (DQML) decoding

An [[n, k]] stabilizer code, denoted by $\mathcal{C}[[n, k]]$, encodes k qubits into a codeword of n qubits. Analogous to classical linear codes, it can be seen as a subspace of dimension $k$ in an $n$-dimensional Hilbert space.

An [[n, k]] stabilizer code can be represented by an $(n-k) \times 2n$ matrix $(Z|X)$ in which its rows $\{\alpha_i \triangleq (z_i, x_i)\}_{i=1}^{n-k}$ satisfy $\alpha_i \odot \alpha_j = 0$, $\forall i, j$, where $\alpha_i \odot \alpha_j \triangleq z_i x_j^T + x_i z_j^T$ [2].

Define the following notation:
1. $\mathcal{S} \triangleq \text{span}(\alpha_1, \alpha_2, \ldots, \alpha_{n-k})$.
2. $\mathcal{L} \triangleq \text{span}(\alpha_{n-k+1}, \ldots, \alpha_n, \beta_{n-k+1}, \beta_n)$ and $\mathcal{F} \triangleq \text{span}(\beta_1, \ldots, \beta_{n-k})$, where $\{\alpha_i\}_{i=n-k+1}^n$ and $\{\beta_j\}_{j=1}^{n-k}$ can be constructed from $\mathcal{S}$ such that [3, Appendix A]
   \[\beta_i \odot \beta_j = 0, \quad \forall i, j; \quad \alpha_i \odot \beta_j = 0, \quad \forall i \neq j, \quad \alpha_i \odot \beta_i = 1, \quad \forall i.\]

3. $\gamma \triangleq \gamma_1 \oplus \gamma_2 \oplus \gamma_3$, where $\gamma_1 \in \mathcal{S}$, $\gamma_2 \in \mathcal{L}$, and $\gamma_3 \triangleq \gamma_3(s) = \sum_{i=1}^{n-k} s_i \beta_i \in \mathcal{F}$ for $s_i \in \{0, 1\}$.

The DQML decoder $g_{\text{DQML}}$ is defined as follows:

$$g_{\text{DQML}}(s) = \arg \max_{\gamma \in \mathcal{F}} \left\{ \sum_{\gamma_1 \in \mathcal{S}} \Pr(\gamma_1 \oplus \gamma_2 \oplus \gamma_3) \right\}$$

2 Main Results

An Expression of the Average Error Probability

Given an arbitrary $(Z|X) = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-k}\}$. Then the exact average success probability based on DQML decoding can be expressed as

$$P_{e, \text{DQML}}(Z|X) = 1 - \sum_{s \in (\mathcal{Z}_2)^{n-k}} \left( \sum_{\gamma_1 \in \mathcal{S}} \Pr(\gamma_1 \oplus \gamma_2 \oplus \gamma_3) \right)$$

1. Let $p$ be the cross-over probability of the Pauli Independent X-Z Channel
2. For $[[n, k]] = [[5, 1]]$, the code achieves the largest minimum distance is the $[[5, 1, 3]]$ perfect code [4], where $[[n, k, d]]$ denotes a quantum stabilizer code with minimum distance $d$.
3. We found that $[[5, 1, 1]]$ code indeed performs better than the famous $[[5, 1, 3]]$ perfect code for certain values of $p$.

Figure 1: $P_{e, \text{DQML}}(\mathcal{C}[[5,1,1]])$ (red curve) and $P_{e, \text{DQML}}(\mathcal{C}[[5,1,3]])$ (blue curve)

References