Project Name:	Optimal Code Design That Minimizes The Maximum-Likelihood Decoding Error Over Classical and Quantum Channels (2/2)
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1 Quantum Stabilizer Codes

In this poster, we investigate the **optimal** $[[n,k]]$ **quantum stabilizer codes** over the Pauli Independent X-Z Channel. By *optimal code* we mean

1. Such code achieves **the smallest exact average error probability** among all possible $[[n,k]]$ quantum stabilizer codes
2. The average error probability is derived using **the quantum maximal-likelihood (DQML) decoding**

An $[[n,k]]$ **stabilizer code, denoted by** $\mathscr{C}^{[[n,k]]}$, encodes k qubits into a codeword of n qubits. Analogous to classical linear codes, it can be seen as a subspace of dimension k in an n-dimensional Hilbert space.

An $[[n,k]]$ **stabilizer code** can be represented by an $(n-k)\times 2n$ **matrix** $(\mathsf{Z}|\mathsf{X})$ in which its rows $\{\boldsymbol{\alpha}_i \triangleq (\boldsymbol{z}_i|\boldsymbol{x}_i)\}_{i=1}^{n-k}$ satisfy $\boldsymbol{\alpha}_i \odot \boldsymbol{\alpha}_j = 0, \ \forall\, i,j$, where $\boldsymbol{\alpha}_i \odot \boldsymbol{\alpha}_j \triangleq \boldsymbol{z}_i\boldsymbol{x}_j^{\mathsf{T}} \oplus \boldsymbol{x}_i\boldsymbol{z}_j^{\mathsf{T}}$ [2].

Define the following notation:

1. $\mathscr{S} \triangleq \operatorname{span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{n-k})$.
2. $\mathscr{L} \triangleq \operatorname{span}(\boldsymbol{\alpha}_{n-k+1}, \ldots, \boldsymbol{\alpha}_n, \boldsymbol{\beta}_{n-k+1}, \boldsymbol{\beta}_n)$ and $\mathscr{T} \triangleq \operatorname{span}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n-k})$, where $\{\boldsymbol{\alpha}_i\}_{i=n-k+1}^{n}$ and $\{\boldsymbol{\beta}_i\}_{j=1}^{n}$ can be constructed from $\mathscr{S}$ such that [3, Appendix A]

$$\begin{aligned} \boldsymbol{\beta}_i \odot \boldsymbol{\beta}_j &= 0, & \forall\, i,j;\\ \boldsymbol{\alpha}_i \odot \boldsymbol{\beta}_j &= 0, & \forall\, i \neq j,\\ \boldsymbol{\alpha}_i \odot \boldsymbol{\beta}_i &= 1, & \forall\, i. \end{aligned}$$

3. $\gamma \triangleq \gamma_1 \oplus \gamma_2 \oplus \gamma_3$, where $\gamma_1 \in \mathscr{S}$, $\gamma_2 \in \mathscr{L}$, and $\gamma_3 \triangleq \gamma_3(\boldsymbol{s}) = \sum_{i=1}^{n-k} s_i \boldsymbol{\beta}_i \in \mathscr{T}$ for $s_i \in \{0,1\}$.

The DQML decoder g_{DQML} is defined as follows:

$$g_{\mathsf{DQML}}(\boldsymbol{s}) = \operatorname*{argmax}_{\gamma_2 \in \mathscr{L}} \left\{ \sum_{\gamma_1 \in \mathscr{S}} \Pr(\gamma_1 \oplus \gamma_2 \oplus \gamma_3) \right\}$$

2 Main Results

An Expression of the Average Error Probability

Given an arbitrary $(\mathsf{Z} \mid \mathsf{X}) = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{n-k}\}$. Then the exact average success probability based on DQML decoding can be expressed as

$$P_{\mathsf{e,DQML}}(\mathsf{Z}|\mathsf{X}) = 1 - \sum_{\substack{\boldsymbol{s}\in(\mathbb{Z}_2)^{n-k}\\ g_{\mathsf{DQML}}(\boldsymbol{s})=\gamma_2^*}} \left(\sum_{\gamma_1 \in \mathscr{S}} \Pr(\gamma_1 \oplus \gamma_2^* \oplus \gamma_3) \right)$$

1. Let p be the cross-over probability of the Pauli Independent X-Z Channel
2. For $[[n,k]] = [[5,1]]$, the code achieves **the largest minimum distance is the** $[[5,1,3]]$ **perfect code** [4], where $[[n,k,d]]$ denotes a quantum stabilizer code with minimum distance d.
3. We found that $[[5,1,1]]$ **code indeed performs better than the famous** $[[5,1,3]]$ **perfect code for certain values of** p

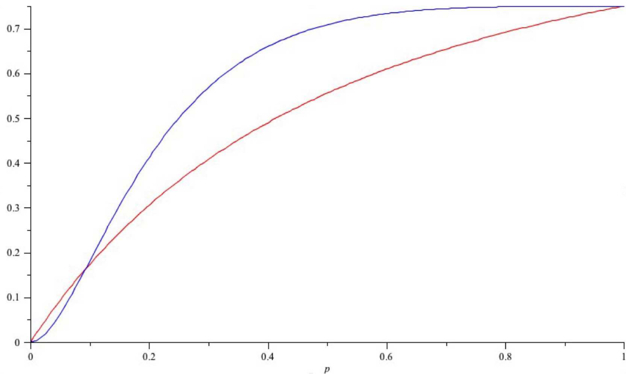

Figure 1: $P_{\mathsf{e,DQML}}(\mathscr{C}^{[[5,1,1]]})$ (red curve) and $P_{\mathsf{e,DQML}}(\mathscr{C}^{[[5,1,3]]})$ (blue curve)

References

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