Project Name:	Optimal Code Design That Minimizes The Maximum-Likelihood Decoding Error Over Classical and Quantum Channels (1/2)
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1 Quantum Stabilizer Codes

In this poster, we introduce a new way of constructing arbitrary quantum stabilizer codes that are analogous to classical linear codes using a **column-wise** approach.

An $[[n, k]]$ **stabilizer code** is a codebook that encodes k qubits into a codeword of n qubits. Analogous to classical linear block codes, a quantum stabilizer code can be described by **a parity check matrix** H **comprising of** $n-k$ **generators** and can be seen as a subspace of size 2^k in an n-dimensional Hilbert space. We illustrate an example for $[[n, k]] = [[3, 1]]$ and define the candidate matrices sets as

$$\mathcal{Q}^{(3,1)} = \Bigg\{ \mathsf{Q}_1^{(3,1)} \triangleq \left(\begin{array}{c|c} 0 & 0 \\ 0 & 0 \end{array}\right), \mathsf{Q}_2^{(3,1)} \triangleq \left(\begin{array}{c|c} 0 & 0 \\ 0 & 1 \end{array}\right), \mathsf{Q}_3^{(3,1)} \triangleq \left(\begin{array}{c|c} 0 & 1 \\ 0 & 0 \end{array}\right), \mathsf{Q}_4^{(3,1)} = \left(\begin{array}{c|c} 0 & 1 \\ 0 & 1 \end{array}\right), \mathsf{Q}_5^{(3,1)} \triangleq \left(\begin{array}{c|c} 0 & 0 \\ 1 & 1 \end{array}\right), \mathsf{Q}_6^{(3,1)} \triangleq \left(\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right), \mathsf{Q}_7^{(3,1)} \triangleq \left(\begin{array}{c|c} 0 & 1 \\ 1 & 1 \end{array}\right), \mathsf{Q}_8^{(3,1)} \triangleq \left(\begin{array}{c|c} 1 & 1 \\ 0 & 0 \end{array}\right), \mathsf{Q}_9^{(3,1)} \triangleq \left(\begin{array}{c|c} 1 & 1 \\ 0 & 1 \end{array}\right), \mathsf{Q}_{10}^{(3,1)} \triangleq \left(\begin{array}{c|c} 1 & 1 \\ 1 & 1 \end{array}\right) \Bigg\}$$

and construct an arbitrary **stabilizer parity-check matrix** as follows:

$$(\mathsf{Z} \mid \mathsf{X}) = \left(\begin{array}{c|c} - \mathbf{z}_1 - & - \mathbf{x}_1 - \\ - \mathbf{z}_2 - & - \mathbf{x}_2 - \end{array}\right) = (\mathsf{A} \mid \mathsf{B}) = \left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ | & | & | & | & | & | \end{array}\right)_{2\times 6},$$

where $(\mathbf{a}_i \mid \mathbf{b}_j) \in \mathcal{Q}^{(3,1)}, \forall 1 \le i, j \le 3$.

Note that each row $\boldsymbol{\alpha}_j \triangleq (\mathbf{z}_i \mid \mathbf{x}_i)$ of (Z|X) must satisfy

$$\boldsymbol{\alpha}_i \odot \boldsymbol{\alpha}_j = 0, \quad \forall i, j;$$

where $\boldsymbol{\alpha}_i \odot \boldsymbol{\alpha}_j \triangleq \mathbf{z}_i \mathbf{x}_j^{\mathsf{T}} \oplus \mathbf{x}_i \mathbf{z}_j^{\mathsf{T}}$. It then corresponds to **a valid quantum stabilizer code** [2]

1. Define $\mathscr{S} \triangleq \operatorname{span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_{n-k})$
2. $\mathscr{L} \triangleq \operatorname{span}(\boldsymbol{\alpha}_{n-k+1}, \dots, \boldsymbol{\alpha}_n, \boldsymbol{\beta}_{n-k+1}, \boldsymbol{\beta}_n)$ and $\mathscr{T} \triangleq \operatorname{span}(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n-k})$ can be constructed from $\mathscr{S}$ such that [3, Appendix A]
$$\boldsymbol{\beta}_i \odot \boldsymbol{\beta}_j = 0, \quad \forall i, j;$$
$$\boldsymbol{\alpha}_i \odot \boldsymbol{\beta}_j = 0, \quad \forall i \neq j;$$
$$\boldsymbol{\alpha}_i \odot \boldsymbol{\beta}_i = 1, \quad \forall i.$$
3. Define an error vector $\boldsymbol{\gamma} \triangleq \boldsymbol{\gamma}_1 \oplus \boldsymbol{\gamma}_2 \oplus \boldsymbol{\gamma}_3$, where $\boldsymbol{\gamma}_1 \in \mathscr{S}$, $\boldsymbol{\gamma}_2 \in \mathscr{L}$, and $\boldsymbol{\gamma}_3 \triangleq \boldsymbol{\gamma}_3(\mathbf{s}) = \sum_{i=1}^{n-k} s_i \boldsymbol{\beta}_i \in \mathscr{T}$ for $s_i \in \{0, 1\}$
4. An error vector is generated by a quantum memoryless channel with probability: $\Pr(\boldsymbol{\gamma})$

2 Main Results

Degenerate Quantum Maximum Likelihood (DQML) Decoder

The DQML decoder g_{DQML} is defined as below:

$$g_{\mathsf{DQML}}(\mathbf{s}) = \operatorname*{argmax}_{\boldsymbol{\gamma}_2 \in \mathscr{L}} \left\{ \sum_{\boldsymbol{\gamma}_1 \in \mathscr{S}} \Pr(\boldsymbol{\gamma}_1 \oplus \boldsymbol{\gamma}_2 \oplus \boldsymbol{\gamma}_3) \right\}$$

Theorem: An Expression for the Exact Average Success Probability

Given an arbitrary $(\mathsf{Z} \mid \mathsf{X}) = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_{n-k}\}$. Then the exact average success probability based on DQML decoding can be expressed as

$$P_{\mathsf{c,DQML}}(\mathsf{Z}|\mathsf{X}) = \sum_{\substack{\mathbf{s} \in (\mathbb{Z}_2)^{n-k} \\ g_{\mathsf{DQML}}(\mathbf{s}) = \boldsymbol{\gamma}_2^*}} \left(\sum_{\boldsymbol{\gamma}_1 \in \mathscr{S}} \Pr(\boldsymbol{\gamma}_1 \oplus \boldsymbol{\gamma}_2^* \oplus \boldsymbol{\gamma}_3) \right)$$

References

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[2] M.-H. Hsieh and F. Le Gall, "NP-hardness of decoding quantum error-correction codes," *Physical Review A*, vol. 83, no. 5, pp. 052331, May 2011.

[3] T. A. Brun, I. Devetak, and M.-H. Hsieh, "Catalytic quantum error correction," *IEEE Trans. Inform. Theory*, vol. 60, no. 6, pp. 3073—3089, Jun. 2014.