A Closed-Form Expression for the Exact Average Error Probability of Arbitrary Binary Codes over the Binary Erasure Channel

Po-Ning Chen and Hsuan-Yin Lin
Department of Electrical and Computer Engineering
National Chiao Tung University (NCTU)
Hsinchu, Taiwan
qponing@mail.nctu.edu.tw, lin.hsuanyin@ieee.org

Stefan M. Moser*
ETH Zurich, Switzerland, and Department of Electrical and Computer Engineering, National Chiao Tung University (NCTU), Hsinchu, Taiwan
stefan.moser@ieee.org

1 General Binary Codes

In this poster, we introduce a new way of constructing arbitrary binary codes (**linear or nonlinear**) using a **column-wise** approach.

A code with M messages and with blocklength n is described by a **code parameter vector $\mathbf{t}$**. We illustrate the function of this code parameter vector with an example for the case of four messages $\mathrm{M} = 4$. We define the candidate columns sets as

$$\mathcal{C}^{(4)} = \left\{ \mathbf{c}_1^{(4)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \mathbf{c}_2^{(4)} \triangleq \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \mathbf{c}_3^{(4)} \triangleq \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \mathbf{c}_4^{(4)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \mathbf{c}_5^{(4)} \triangleq \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \mathbf{c}_6^{(4)} \triangleq \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \mathbf{c}_7^{(4)} \triangleq \begin{pmatrix}0\\1\\1\\1\end{pmatrix} \right\}$$

and describe the a code as follows:

1. Let t_j denote the number of the corresponding candidate columns $\mathbf{c}_j^{(4)}$ appearing in the codebook matrix of $\mathscr{C}^{(4,n)}$, $j = 1, \ldots, 7$.
2. Consider any binary code with blocklength n by code parameters vector $\mathbf{t}$:

$$n = \sum_{j=1}^{7} t_j \quad \text{where } \mathbf{t} = [t_1, t_2, \ldots, t_7]$$

A codebook $\mathscr{C}_{\mathbf{t}}^{(4,7)}$ of type $\mathbf{t} = [2, 0, 2, 0, 2, 1, 0]$ is equivalent to all the columns permutations of the following codebook:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

2 Main Results

Definition: r-wise Hamming Distance

Given any (M, n) binary codebook $\mathscr{C}^{(\mathrm{M},n)}$, we could always denote it by a specific code parameters vector $\mathbf{t}$. Choose some $1 \le i_1 < i_2 < \cdots < i_r \le \mathrm{M}$, $2 \le r \le \mathrm{M}$, the r-wise Hamming distances $d_{i_1 i_2 \cdots i_r}^{(\mathrm{M},n)}$ are the number of columns such that those $i_1, i_2, \ldots, i_r$th components are not equal. For the case of $\mathrm{M} = 4$, we have

$$\begin{aligned}
d_{12}^{(4,n)} &= n - (t_1 + t_2 + t_3) & d_{123}^{(4,n)} &= n - t_1 \\
d_{13}^{(4,n)} &= n - (t_1 + t_4 + t_5) & d_{124}^{(4,n)} &= n - t_2 \\
d_{14}^{(4,n)} &= n - (t_1 + t_6 + t_7) & d_{134}^{(4,n)} &= n - t_4 \\
d_{23}^{(4,n)} &= n - (t_2 + t_4 + t_6) & d_{234}^{(4,n)} &= n - t_7 \\
d_{24}^{(4,n)} &= n - (t_2 + t_5 + t_7) & d_{1234}^{(4,n)} &= n \\
d_{34}^{(4,n)} &= n - (t_3 + t_4 + t_7)
\end{aligned}$$

Theorem: A closed-form expression for the Exact Average Error Probability

Consider a BEC with arbitrary erasure probability $0 \le \delta < 1$ and an arbitrary code $\mathscr{C}_{\mathbf{t}}^{(\mathrm{M},n)}$. The average ML error probability is a function of $\mathbf{t}$ as follows:

$$P_{\mathrm{e}}\left(\mathscr{C}_{\mathbf{t}}^{(\mathrm{M},n)}\right) = \frac{1}{\mathrm{M}} \sum_{r=2}^{\mathrm{M}} (-1)^r \sum_{\substack{\mathcal{I} \subseteq \{1,\ldots,\mathrm{M}\} \\ |\mathcal{I}| = r}} \delta^{d_{\mathcal{I}}^{(\mathrm{M},n)}}.$$

References

[1] P.-N. Chen, H.-Y. Lin, and S. M. Moser, "Weak flip codes and applications to optimal code design on the binary erasure channel," in Proc. 50th Allerton Conf. Commun., Contr. and Comput., Monticello, IL, USA, Oct. 1–5, 2012.

[2] P.-N. Chen, H.-Y. Lin, and S. M. Moser, "Optimal ultrasmall block-codes for binary discrete memoryless channels," IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 7346–7378, Nov. 2013.